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On the spectral analysis of trajectories in near-integrable Hamiltonian systems

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Abstract. Spectral properties associated with the deformation of tori and the transition to chaos in near-integrable Hamiltonian systems are studied. Information about the construction of tori is provided by studying the evolution of the integrals of the unperturbed system when a perturbation is added. We show that the low band of the power spectrum converges exponentially for regular trajectories but we pass abruptly to $1/f^{\alpha}$ divergence when chaos occurs. These results are valid for systems of two or more degrees of freedom and provide a clear distinction between regular and chaotic motion.

1. Introduction

Power spectral analysis has often been applied to the study of nonlinear dynamical systems. Multifrequency dynamical variables which exhibit quasiperiodic or periodic motion are characterized by few major sharp peaks in their spectrum located at integer combinations of the fundamental frequencies, in contrast to the broad features which emerge in the case of turbulence and chaos [1-4]. The transition to chaos by the period doubling route can be studied by inspecting the birth of peaks in the spectrum and their position in relation to the initial frequencies (e.g. see [5]) while intermittency, which arises in classical Hamiltonian systems, results in an incoherent superposition of frequencies [6]. Power spectral analysis is also one of the most useful techniques for semiclassical or quantum studies of molecular dynamics where molecular spectra are obtained by classical Hamiltonian trajectories [7-9].

A regular (quasiperiodic) trajectory has a power spectrum of the form [8]

$$P(\Omega) = \sum_{m} |x_m|^2 \delta(m\omega - \Omega) \tag{1}$$

where $m = (m_1, \ldots, m_n) \in Z^n$ and ω is the vector of the fundamental frequencies. From (1) it is evident that the power spectrum consists of a set of discrete peaks which appear at positions $\Omega = m\omega$. Only a few of them are of significant amplitude; however, they are distinct and denote that the trajectory is regular. In the case of a chaotic trajectory the spectrum becomes irregular and shows a continuous distribution of peaks [10-12]. It has also been shown that in some cases chaotic trajectories of Hamiltonian systems exhibit $1/f^{\alpha}$ noise with $a \approx 1$ [13, 14] (f corresponds to Ω in our notation), and a statistical description of this phenomenon has been given for systems of two degrees of freedom.

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In the present paper we study from a classical point of view the distribution of the low frequencies for regular trajectories which lie on invariant tori of dimension n. We show that $P(\Omega)$ converges exponentially to zero as $\Omega \to 0$ but, in a chaotic region, the power spectrum lacks such a convergence. Numerical results show that the spectrum either saturates to a finite value for $\Omega = 0$ or diverges as $1/f^{\alpha}$. We consider near-integrable systems which are described by Hamiltonians of the form

$$H(p, q) = H_0(p, q) + \varepsilon H_1(p, q) \tag{2}$$

where H_0 is the integrable part and εH_1 denotes the perturbation. Although the behaviour of such systems with $n = 2$ degrees of freedom has been studied in detail, for systems of $n > 2$ there are many open questions. KAM theorem ensures that under small perturbations most of the tori, where quasiperiodic trajectories lie, persist even for $n \geq 3$ but the mechanism of the break-up of resonant tori and the topological structure of the phase space is not completely understood [15]. The aim of this paper is to extract spectral properties of trajectories in such systems which can provide some information about the construction of tori and their break-up. Our study is based on the Fourier spectral analysis of functions $I_j = I_j(q, p)$ which are integrals of motion for the integrable part H_0. Since these functions are constants in the unperturbed system during time evolution on a particular trajectory, their variation when a perturbation is added is related exclusively to the deformation of tori under the perturbation. We show that the power spectrum of I_j is related to the effect of small denominators on the construction of a torus. Its behaviour as $\Omega \to 0$ allows a sharp distinction between regular and chaotic motion even for systems of many degrees of freedom and provides some information about the transition from the first type of motion to the second.

2. Variation of the integrals and spectral analysis

We consider near-integrable Hamiltonian systems of the form (2) with ε small. The integrable part $H_0(p, q)$ possesses n integrals $I_j(p, q)$ in involution, i.e. the following conditions are satisfied:

$$[I_j, H_0] = 0 \qquad [I_j, I_i] = 0 \qquad \forall j, i = 1, \ldots, n \tag{3}$$

where [,] denotes the Poisson bracket.

For $\varepsilon = 0$, each I_j assumes a constant value c_j on a specific trajectory. The constants c_j label the torus where the trajectory lies, i.e. every torus corresponds to a vector $c = (c_1, c_2, \ldots, c_n)$ by an injection. For a small but non-zero ε, I_j are, in general, no longer constants (except for the perturbed Hamiltonian itself) but vary in time according to the equation

$$\mathrm{d}I_j/\mathrm{d}t = [I_j, H] = \varepsilon[I_j, H_1]_0 + \mathrm{O}(\varepsilon^2) \tag{4}$$

where the subscript 0 denotes that the corresponding quantity is computed in the unperturbed system. For time intervals of the order of 1, I_j vary slowly and this variation is of the order of ε, providing the possibility to apply averaging methods [16–18]. However, the variation of I_j may become considerable for long time intervals ($t > 1/\varepsilon$). For the case of $n = 2$, the tori divide the three-dimensional level manifold where the trajectories lie. Therefore, since for small ε the deformation of most tori is very small, $I_j = I_j(t)$ vary little over an infinite time interval. For $n > 2$ the tori do not divide the level manifold and, in this case, Nekhorosev's theorem [19] provides an exponential estimation for the time interval in which the variation of the integrals in the perturbed system is small even inside thin stochastic layers.

Next we consider as such integrals the actions of H_0 and write the Hamiltonian (2) in action-angle variables I, ϕ, in the form

$$H = H_0(I) + \varepsilon H_1(I, \phi) + O(\varepsilon^2) \tag{5}$$

where H_1 is mod 2π with respect to ϕ. The equations of motion up to terms $O(\varepsilon)$ are

$$\dot{I}_j = -\varepsilon \partial H_1 / \partial \phi_j \tag{6a}$$

$$\dot{\phi}_j = \omega_j(I) + \varepsilon \partial H_1 / \partial I_j. \tag{6b}$$

H_1 can be expanded in multiple Fourier series with respect to ϕ:

$$H_1 = \sum_m h_m(I) \exp(\mathrm{i} m\phi). \tag{7}$$

By substituting (7) in the equations of motion and integrating we obtain [20]

$$I_j = I_{j0} + \varepsilon \sum_m \frac{m_j h_m}{m\omega} \exp(\mathrm{i} m\omega t) + O(\varepsilon^2). \tag{8}$$

The above expression cannot be considered an asymptotic solution for the variation of I_j, since it contains small denominators and its convergence is doubtful. This problem is also present in the power spectrum of I_j, which has the form

$$P_j(\Omega) = \sum_m \left| \varepsilon \frac{m_j h_m}{m\omega} \right|^2 \delta(m\omega - \Omega) + O(\varepsilon^3). \tag{9}$$

The denominator $m\omega$ may take arbitrarily small values so, from a first examination, significant amplitudes at very low frequencies will appear. The terms of higher order also contain such denominators and may play an important role in the construction of the spectrum. Since $I_j(t)$ are smooth functions, $P_j(\Omega)$ converge to zero at high frequencies ($\Omega \to \infty$). The structure of the power spectrum depends on the values which the terms $h_m/m\omega$ may take. Their significance is apparent in classical perturbation techniques. Consider, for example, the sequence of transformations [21]

$S^1(I', \phi): I, \phi \to I', \phi'$ such that $H(I, \phi) = H'(I') + \varepsilon^2 H_2(I', \phi')$

$S^2(I'', \phi'): I', \phi' \to I'', \phi''$ such that $H(I, \phi) = H''(I'') + \varepsilon^3 H_3(I'', \phi'')$

etc. If we take $S^1 = I'\phi + \varepsilon S_1^1$ and expand S_1^1 in multiple Fourier series with respect to ϕ, its coefficients will be $S_m^1 = \mathrm{i} h_m / m\omega$. In the same way, similar terms S_m^n will appear in the higher order terms $O(\varepsilon^n)$ of the spectrum. So with this procedure new tori are constructed for the perturbed trajectories which will support quasiperiodic motion with frequencies close to the unperturbed ones. The validity of this procedure depends, however, on the convergence of the series for every generating function S^i and on the convergence of the sequence of transformations. We can observe that the behaviour of the sequence of S^i is reflected by the structure of the power spectrum (9).

2.1. *Power spectrum on invariant tori*

According to KAM theory most of the invariant tori of the unperturbed system persist under the perturbation and are only slightly deformed. The trajectory on such a torus is quasiperiodic and the frequency vector ω satisfies the condition [21]

$$|m\omega| \geqslant k|m|^{-(n+1)} \qquad |m| = |m_1| + |m_2| + \ldots + |m_n| \tag{10}$$

where k is a fixed positive constant. Also, if H_1 is analytic, the Fourier coefficients h_m decrease exponentially with $|m|$, i.e. the following condition holds:

$$|h_m| \leqslant M \exp(-|m|\rho) \tag{11}$$

with $\rho > 0$ and $M > 0$ such that $|H_1| < M$. By taking into account the above inequalities, it can be proved that the terms $|h_m/m\omega|$ decrease rapidly in geometric progression [21]. The same behaviour is true for the terms $|m_j h_m/m\omega|$ since m_j increases linearly. The convergence of the sequence with respect to ε^n is also guaranteed by Kolmogorov's quadratic convergence. So there are terms in (9) at frequencies $\Omega = m\omega$ which may approach zero for large $|m|$, but this implies that the corresponding $P_j(\Omega)$ converge rapidly to zero.

An estimation about the convergence of the power spectrum as $\Omega \to 0$ and the distribution of the frequencies can be found by keeping in (10) only the terms of the order of ε^2. We define the 'frequency set' $\Gamma_\omega = \{\Omega \mid \Omega = m\omega,\ m \in Z^n\} \subset \boldsymbol{R}$, where ω is the constant frequency vector of a particular torus with incommensurable components. The map $f: m \in Z^n \to \Omega \in \Gamma_\omega$ is a bijection so we define the continuous function $A(\Omega)$ in Γ, which is related to $P_j(\Omega)$, as follows:

$$A(\Omega) = \begin{cases} |\varepsilon m_j h_m/\Omega|^2 & \Omega \in \Gamma \\ 0 & \Omega \notin \Gamma \end{cases} \tag{12}$$

where m is the image of Ω under the map f^{-1}.

By taking into account (11) and that $|m| \geqslant |m_j|$, we obtain

$$A(\Omega) \leqslant \left(\varepsilon M \frac{|m| \exp(-|m|\rho)}{\Omega} \right)^2.$$

Since the exponential function tends faster to infinity than a linear one, we can find an $m_0(\delta > 0)$ such that $|m| < \exp(|m|\delta)$, $\forall |m| > |m_0|$. We can select $0 < \delta < \rho$ and also those m with $|m| > |m_0|$ such that $0 < |m\omega| < |m_0\omega|$ or equivalently $0 < \Omega < \Omega_0$. So in the frequency domain $(0, \Omega_0)$ we have

$$A(\Omega) \leqslant \left(\varepsilon M \frac{\exp(-|m|(\rho - \delta))}{\Omega} \right)^2.$$

From (10) we take $|m| \geqslant (k/\Omega)^{1+n}$ and by substituting in the above expression we finally get

$$A(\Omega) \leqslant \left(\varepsilon M \frac{\exp(-a\Omega^{-1/p})}{\Omega} \right)^2 \tag{13}$$

where $p = n + 1$ and $a = k^{1/p}(\rho - \delta) > 0$. Since $\mu = \Omega^{1/p} \to 0$ and $\exp(-1/\mu) = o(\mu^q)$ for $\Omega \to 0$ and for all $q \in \boldsymbol{N}$ [22], the above relation guarantees an exponential convergence to zero for the low-frequency band of the power spectrum. The rate of the convergence is related to the parameter k, which depends on the nearby resonance and the number of degrees of freedom.

We mentioned above that $P_j(\Omega)$ converges to zero for high frequencies, so the power spectrum is bounded in the whole frequency domain. By taking into account inequalities (10) and (11) and since $1 > k > \varepsilon^{1/2}$ [16] we get

$$A(\Omega) \leqslant |\sqrt{\varepsilon}\, M\, \mathrm{e}^{-|m|\rho} |m|^{p+1}|^2 = |\sqrt{\varepsilon}\, ML(|m|)|^2. \tag{14}$$

The function $L: \boldsymbol{Z}^+ \to \boldsymbol{R}^+$ has an upper bound L_0. If we assume L_0 and M to be of the order of 1, we conclude that $A(\Omega)$ does not exceed $O(\varepsilon)$.

It is obvious from the definition of $A(\Omega)$ that if $\Omega = O(1)$ then $A(\Omega) \leqslant O(\varepsilon^2)$. So significant peaks may appear only for low frequencies $\Omega_l = l\omega = O(\varepsilon)$ with $|l|$ small. Such low frequencies may appear near a resonant torus where $k_1/\omega_1 = k_2/\omega_2 = \ldots = k_n/\omega_n$, or equivalently $n-1$ independent relations of the form $m^i\omega = 0$, $i = 1, \ldots, n-1$, are satisfied. In the neighbourhood of a resonance, we write the previous relations as $m^i\omega = b_i\varepsilon$, $b_i = O(1) \in R$, so there exist terms in (9) at $n-1$ low frequencies of $O(\varepsilon)$ and their linear combinations. No other frequencies with significant amplitudes may appear and this can be shown as follows. Consider a small frequency $\Omega_l = l\omega = b\varepsilon$, where $b = O(1)$ and l is an integer vector linearly independent of all m^i. We form the system

$$\begin{pmatrix} m_1^1 & m_2^1 & \ldots & m_n^1 \\ m_1^2 & m_2^2 & \ldots & m_n^2 \\ \ldots & \ldots & \ldots & \ldots \\ m_1^{n-1} & m_2^{n-1} & \ldots & m_n^{n-1} \\ l_1 & l_2 & \ldots & l_n \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_{n-1} \\ \omega_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b \end{pmatrix} \varepsilon \qquad \text{or} \qquad M\omega = B\varepsilon.$$

Since M is a non-singular integer matrix, with $|\det(M)| \geqslant 1$, we can solve with respect to ω:

$$\omega = \frac{\operatorname{adj}(M)}{\det(M)} \begin{pmatrix} b_1 \\ \vdots \\ b \end{pmatrix} \varepsilon. \tag{15}$$

The frequencies ω of the unperturbed torus are of the order of 1 so the elements of $\operatorname{adj}(M)$, which are combinations of integer numbers, should be of the order of $1/\varepsilon$. Under the assumption that m_j^i are small integers, we conclude that the components of l must be of the order of $1/\varepsilon$, i.e. they are very large integers, and consequently the corresponding peak at $\Omega_l = l\omega$ has negligible amplitude.

So the evolution of I_j when the motion takes place on an invariant torus does not exhibit very slow oscillations with significant amplitudes. Numerical results show that the spectrum of I_j on a deformed invariant torus contains only a few main peaks but is enriched when a resonance is approached.

2.2. *Power spectrum of I and break-up of tori*

If the perturbation in (2) is very small but non-zero ($\varepsilon \ll 1$) the phase space is mostly filled by invariant tori but regions of exponentially small measure also exist around the destroyed resonant tori of the unperturbed system where the motion is chaotic.

In the case $n = 2$, chaotic regions may become considerable by increasing ε. The break-up of a resonant torus leads, according to Poincaré–Birkhoff theorem, to the formation on the Poincaré section of a chain of islands and a thin stochastic zone around the homoclinic webs associated with the hyperbolic periodic orbits. The motion in the islands is still regular, but one additional frequency appears corresponding to a spiral motion around the unperturbed trajectory. The power spectrum of I on a chain of islands is affected by the appearance of new peaks, whose frequency and amplitude depend on the number and the order of the islands [23]. Subsequent break-up of the islands leads to the formation of higher-order smaller islands and consequently in the appearance of new subsequent frequencies in the spectrum. The power spectrum is thus enriched and its structure becomes more complicated compared to the spectrum along an invariant curve, but it is still discrete and no significant peaks appear as $\Omega \to 0$.

In the stochastic region, the trajectories evolve in a random fashion. The long time which these trajectories spend in the vicinity of a hyperbolic fixed point results in the appearance of low frequencies in the spectrum. It is conjectured that the spectrum of a chaotic trajectory corresponds to a continuous set of frequencies. The analysis of Noid *et al* [8–11] and Powell [12] exploits mainly this difference, i.e. regular orbits have discrete spectra in contrast to chaotic trajectories, which have a continuous spectrum. On the other hand, Geisel *et al* [13] showed the existence of $1/f$ noise for Hamiltonian chaotic trajectories. The power spectrum of the integrals of motion of the unperturbed system also exhibits this feature, which may be a consequence of the divergence of the classical perturbation series. The denominators in the expansion (8) may take arbitrarily small values since the condition (10) does not hold near a resonance. So an irregular continuous distribution of peaks with significant amplitudes appear in the low frequency band, which suggests an irregular evolution of I_j indicating that the trajectory does not lie on a torus.

3. Examples

We demonstrate the previous concepts by applying spectral analysis on the integrals I_j first to a non-homogeneous quartic potential of $n=2$ for purposes of comparison, and secondly to an extension of the previous model to $n=3$ which describes a system of three nonlinear oscillators under a weak coupling.

Since we focus our interest on the low-frequency domain, numerical integration of Hamilton's equations must be performed over long time intervals. For the following examples the particular trajectories were integrated for approximately 8000 time units using double precision. For the construction of the power spectra we used FFT routines with 4096 sample data analysis selected by interpolation from a periodic-like part of $I(t)$ and applying Hanning filter [12]. Since noise, wrap-around pollution and leakage cannot be avoided completely, the exponential convergence of the low band may not be apparent, but the lack or the presence of significant peaks in this region is enough to reveal information on how an invariant torus is affected by the perturbation. A log–log scale has been used for all spectra illustrated, in order to emphasize the details of the low-frequency domain.

3.1. Two nonlinear coupled oscillators

As an example of a system with two degrees of freedom we consider the planar Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+\tfrac{1}{4}(x+y)^2+\tfrac{1}{4}(x-ky)^4 \tag{16}$$

which is integrable for $k=1$ (separates by a $\pi/4$ rotation), the second invariant being

$$I=(p_x+p_y)^2+(x+y)^2 \tag{17}$$

while $\varepsilon=1-k$ is the perturbation parameter. We selected this system as a particularly simple one, which at the same time possesses on a single Poincaré section most of the main features found in a generic Hamiltonian system with $n=2$.

For $\varepsilon=0$, the existence of two stable periodic orbits at $x=0$, $p_x=\pm0.5$ can be shown. These periodic orbits are continued for $\varepsilon>0$ and appear as fixed points surrounded by invariant curves on the section. We study the region around the upper fixed point ($x = 0$, $p_x = 0.5$) where an invariant curve which corresponds to 1:3 resonance exists. This curve breaks down under the perturbation, giving rise to a Poincaré–Birkhoff chain of three stable fixed points surrounded by first-order islands and a chaotic layer along the homoclinic connections of the three hyperbolic fixed points. An enlargement of this area for $\varepsilon = 0.03$ is shown in figure 1.

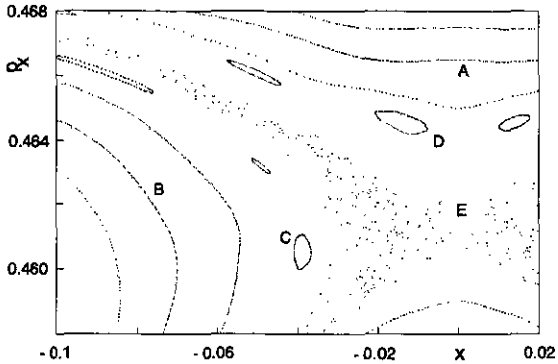

Figure 1. Poincaré section around a hyperbolic fixed point of the 1:3 resonance. Letters denote the trajectories where the corresponding power spectra have been obtained.

The power spectrum of I on an invariant curve near the periodic orbit, which remains stable under the perturbation, is shown in figure 2(a). We observe only a few major peaks at high frequencies. The non-existence of peaks at low frequencies indicates that the tori around the periodic orbit have been deformed slightly and may persist for larger perturbations (in this case the particular torus breaks up for $\varepsilon > 0.15$). This is also a typical power spectrum for deformed invariant curves far from significant resonances. The invariant curve A (figure 1) is near the 1:3 resonance and, as was expected, this torus is subjected to a larger deformation under the perturbation. The spectrum of I in figure 2(b) shows a greater number of major peaks at high frequencies, in comparison to the preceding case, but the most interesting feature is the appearance of many peaks to the right of the spectrum, which means that the corresponding torus has been deformed remarkably under the perturbation.

In figure 3(a) the power spectrum on the island chain B is shown. This is a first-order Poincaré–Birkhoff chain at the 1:3 resonance. The same significant peaks as in the previous case are present, followed by the generation of new peaks around them. A sequence of peaks as $\Omega \to 0$ also arises, indicating a substantial deformation. The island C belongs to a second-order chain of four islands formed around the stable fixed points at the 1:3 resonance. The power spectrum (figure 3(b)) is enriched by the generation of additional notable peaks near every original one. The low band has been filled by some significant peaks which show an irregular distribution; however, the spectrum seems to converge as $\Omega \to 0$, indicating a regular trajectory.

In figure 3(c) the spectrum of I on the twin chain D at the 3:10 resonance is shown. Twin chains are formed when the twist condition of the Poincaré–Birkhoff theorem does not hold and the rotation number presents an extremum along the invariant curves of the integrable system [24, 25]. The spectrum is rich but it still appears to be discrete. A major peak is apparent near zero but the low-frequency band is not filled by considerable peaks.

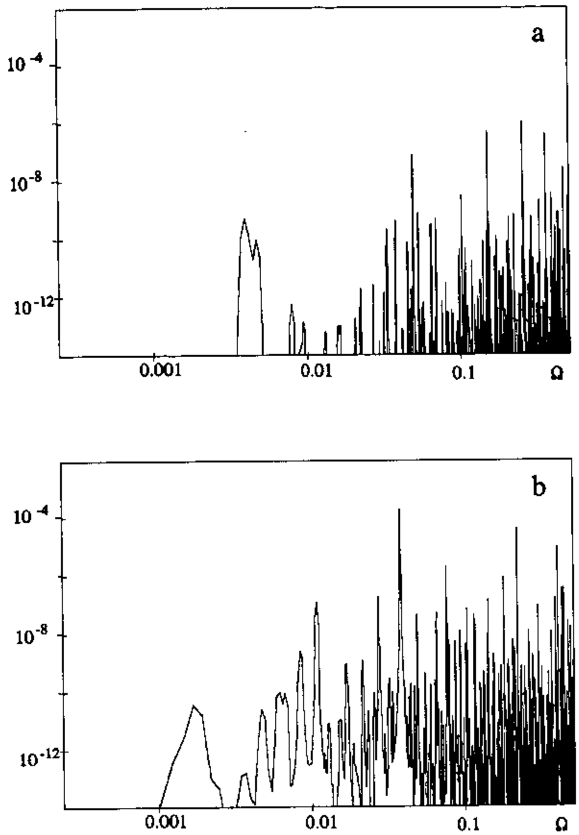

Figure 2. Power spectra for trajectories (*a*) near the periodic orbit and (*b*) near the 1:3 resonance (trajectory A in figure 1).

Finally, in figure 3(*d*) we present the power spectrum of I on a trajectory E in the chaotic region along the homoclinic connections in the 1:3 resonance. The qualitatively different features are obvious. The spectrum seems to be continuous but the most characteristic feature is the divergence at the low frequencies where a broad series of peaks of significant amplitude arises as the frequency tends to zero. This feature, which cannot appear in a regular orbit as has been shown in the preceding section, is peculiar to chaotic motion and is apparent in the spectrum of I along all chaotic regions independently of their width in the phase space.

3.2. Three nonlinear coupled oscillators

Next we consider an extension of the Hamiltonian (16) in three degrees of freedom, as follows:

$$H=\tfrac{1}{2}(p_x^2+p_y^2+p_z^2)+\tfrac{1}{4}(x+y)^2+\frac{A}{2}z^2+\tfrac{1}{4}(x-ky)^4+\tfrac{1}{4}z^4-\varepsilon xyz^2 \qquad (18)$$

which for $k=1$ and $\varepsilon=0$ possesses two invariants, namely I_1 given by (17) and

$$I_2=p_z^2+Az^2+\tfrac{1}{2}z^4. \qquad (19)$$

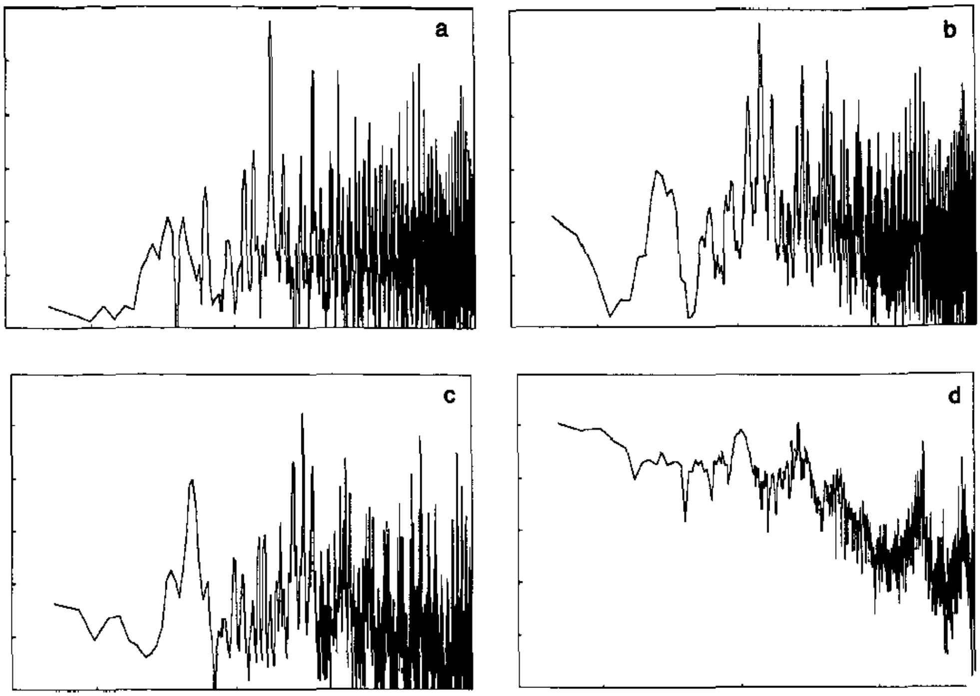

Figure 3. Power spectra with the same scale as in figure 2 for (a) a first-order island (B), (b) a second-order island (C), (c) a twin chain (D) and (d) a chaotic trajectory (E).

We take $k=1$ and we restrict ourselves to the effect of the perturbation εxyz^2 on two regions of the phase space, first close to the periodic orbit of (16) and secondly to the region near the resonance 1 : 3. For the first case and for values of ε up to 0.3 stochasticity has not been observed for $A>0$. The trajectories seem to lie on invariant tori which persist for large perturbations. But for $A<0$, where the motion in z becomes unstable, the spectra of I_1 and I_2 indicate the existence of chaotic motion for $\varepsilon>0$. In the second case chaotic motion appears also for $A>0$ when ε becomes greater than 0.22 (figure 4). In every case we selected as initial conditions for the vertical motion $z=0.1$ and $p_z=0$.

The formation of a thin chaotic zone results in a remarkable change in the low band, in contrast to the region of high frequencies, the form of which seems to be more or less unaffected. In order to illustrate how the low band is affected we calculate the quantity P_{lb}, which is the power of the low-frequency band per total power of the spectrum. There is not a strict definition of the upper limit on this low band. For our case we consider as such $\frac{1}{100}$ of the total spectrum. In figure 5 the variation of this quantity with respect to the parameters of the Hamiltonian (18) is shown. For the case near the resonance 1 : 3, figure 5(a) shows an abrupt increase of the low-band power for $\varepsilon=0.22$ where a transition from regular to chaotic motion occurs. The same behaviour is also observed in figure 5(b), which corresponds to initial conditions close to the periodic trajectory for $\varepsilon=0.1$. For $A>0$, z exhibits oscillatory motion and the spectra of I_1 and I_2 for the tori in the region around the periodic orbit are characterized by a lack of significant peaks at low frequencies. These tori also persist for small negative values of A, but for $A<-0.1$ the instability which appears in the motion of z leads to the generation of chaotic motion.

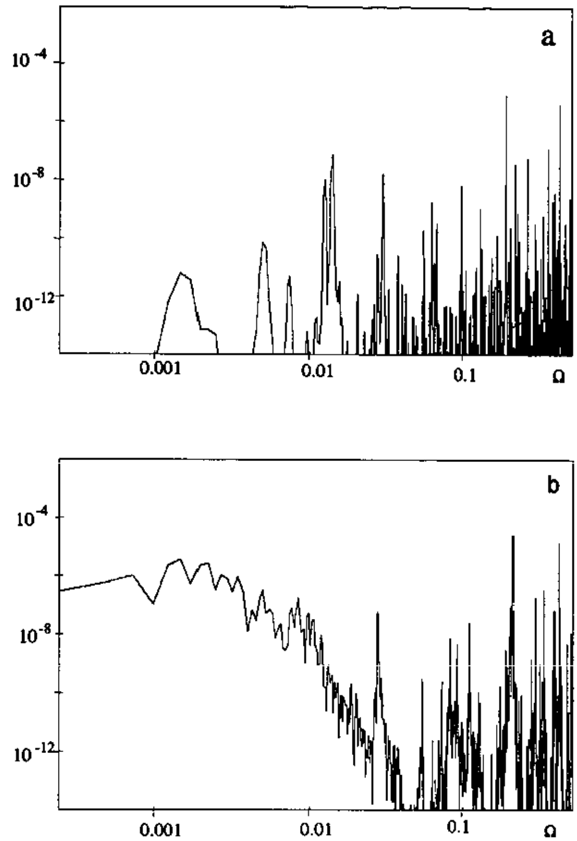

Figure 4. Power spectra for a Hamiltonian with three degrees of freedom with $A = 0.5$. (a) For $\varepsilon = 0.15$ the convergence of spectrum as $\Omega \to 0$ indicates the existence of an invariant torus. (b) For $\varepsilon = 0.25$ the spectrum does not converge to zero as $\Omega \to 0$, indicating chaotic motion.

Thus the qualitative features of the low band of the spectrum of I_j seem to be highly indicative for the distinction between regular and chaotic trajectories. The transition from one kind of motion to the other happens abruptly and from an exponential convergence we pass to a divergence such that $P(\Omega)$ levels off to a finite value at $\Omega = 0$ and indicates a normal diffusion process characterized by a finite diffusion coefficient, or is similar to $1/f^{\alpha}$ noise [13]. These two different situations have been obtained in the cases examined above. For $A > 0$ (figure 4(a)) the spectrum seems to converge at a finite value as Ω tends to zero but for $A < 0$ the phenomenon of $1/f$ noise is apparent (figure 6).

4. Conclusions

The surface of section method provides a powerful tool for obtaining information about the nature of motion in Hamiltonian systems of two degrees of freedom and in these systems one can sharply distinguish between regular and chaotic motion. This method loses much of its strength when one considers systems of three degrees of

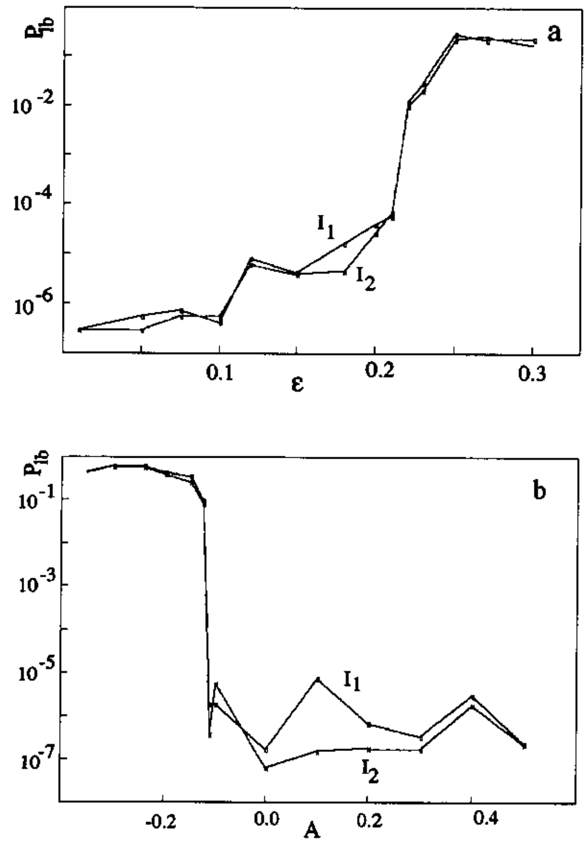

Figure 5. The ratio of the power of the low band per total power. (*a*) For $A=\frac{1}{2}$ versus ε. (*b*) For $\varepsilon=0.1$ versus A. An abrupt increase occurs at the transition from regular to chaotic motion.

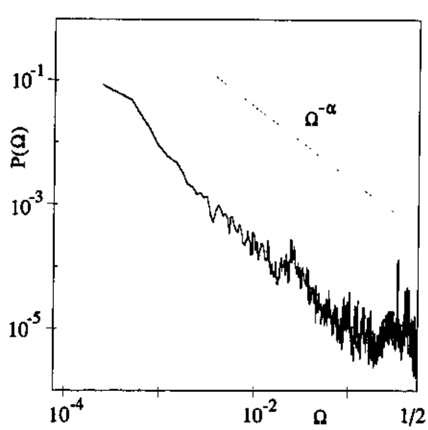

Figure 6. $1/f^{\alpha}$ noise, with $a\approx1.3$, for a chaotic trajectory with $A=-0.3$ and $\varepsilon=0.1$. The spectrum has been averaged over 20 samples, each one having a length of 4096 points.

freedom or more, since in these cases the corresponding section is at least four-dimensional. By computing the largest Lyapunov exponent one may still distinguish chaotic from regular motion, but this method is time-consuming and, in multi-dimensional systems, is affected by diffusion processes.

In this paper spectral properties associated with the deformation of tori and the transition to chaos in near-integrable Hamiltonian systems have been studied. Our method is based on the spectral analysis of the integrals of motion of the integrable part along regular or chaotic trajectories in the perturbed system. A connection of their power spectrum to the small denominators of classical perturbation theory has been shown. Under some necessary conditions for the existence of invariant tori, an estimation of the low band of the spectrum has been obtained which illustrates the rapid convergence of the power spectrum to zero as $\Omega \to 0$. So quasiperiodic trajectories are distinguished from chaotic ones whose power spectrum is characterized by an irregular continuous distribution of significant peaks at low frequencies.

Since the integrals I_j are constants in the unperturbed system, the power spectrum implies information exclusively about the deformation of tori under the perturbation. This deformation is reflected by the number of spectral peaks and their position and amplitude. Tori near a significant resonance are strongly deformed and the power spectrum consists of a set of major peaks, in contrast to robust invariant tori which exhibit a neat spectrum with insignificant peaks at low frequencies. In the case of chaos the low band exhibits a completely different behaviour which is characterized by generation of an irregular continuous distribution of frequencies with amplitudes of the same order of magnitude as the major high-frequency peaks, or greater.

The significance of the low band of the spectrum has also been noticed by Ostlund *et al* [1, 2] for two-dimensional dissipative systems where the structure of the low band seems to be repeated through the whole spectrum and is universal. Similar properties may also be true for the power spectrum of the integrals I_j since in both cases the construction of the spectrum is based on the integer combinations of the fundamental frequencies of the system. On the other hand, Geisel *et al* [13, 14] showed that the trapping of chaotic orbits in a self-similar hierarchy of nested islands in Hamiltonian systems of two degrees of freedom may be responsible for the $1/f^{\alpha}$ divergence in the low band of the spectrum of a dynamical variable on these orbits. Our results also indicate in some cases $1/f^{\alpha}$ divergence and it is an open question if this universal phenomenon [26] may be related to chaos in Hamiltonian systems of more than two degrees of freedom. It is also worth noting the abrupt transition from the exponential convergence to the $1/f^{\alpha}$ distribution. The expansion (9) of the power spectrum cannot provide an explanation for such a transition since the higher-order terms in this expansion diverge near resonances where chaotic motions occur. The slope a is related to the diffusion of the mean square displacement [14] but there are also some other points which need further investigation, for example the relation of the rate of divergence or the power of the low band to the largest Lyapunov exponent and the effect of diffusion processes on the power spectrum of I. Recent results by Sepúlveda *et al* [27] show an apparent connection between the average exponential separation of orbits in the standard map and the continuous background of their spectrum. In our examples, such a continuous background is very evident in the low band of the spectrum for chaotic orbits, while there is no important qualitative change in the high-frequency band as we pass from a regular to a weakly chaotic orbit.

As can be found from inequality (13), the exponential convergence of the spectrum is valid for $\Omega < \Omega^*$ where Ω^* depends on the number n of degrees of freedom as

$(n+1)^{-(n+1)}$, so that Ω^* becomes very small for a large number of degrees of freedom. In this case, this exponential convergence cannot be detected and no information on the energy transport among the various degrees of freedom in a multidimensional system can be obtained (e.g. [28]). It would be interesting, however, to relate the divergence of the low band to the energy transport in a system of a few degrees of freedom (e.g. [29]).

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